Magnetic Monopoles and Gauge Theory

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I

A *magnetic monopole* may be defined as a region of space, which can be enclosed by a two-dimensional surface S, such that the total flux of magnetism through S is non-zero. If B is a three-dimensional vector representing the magnetic field, and $\underset{\sim}{n}$ is the unit normal vector on S, then we may express this statement by the equation $\iint \underset{\sim}{B} \cdot \underset{\sim}{n}\, dS \neq 0$, where the integral extends over S. More loosely, one may conceive of a magnetic monopole as a particle endowed with a "magnetic charge", by anology with electric charge.

In classical physics, the electric and magnetic fields are governed by a set of partial differential equations due to Maxwell. One of these equations is essentially equivalent to the statement that the above double integer should vanish: that is, the Maxwell equations imply that magnetic monopoles do not exist. Indeed, with a few controversial exceptions, no observations of such particles have yet been made. However, it is to be emphasized that the Maxwell equations merely *express* in mathematical form the condition that magnetic monopoles do not exist, and that it is a trivial matter to modify them (in a way which affect no other prediction of the theory) so that magnetic charge is in fact permitted. Thus, the non-existence of magnetic monopoles cannot truly be regarded as a prediction of the Maxwell theory-on the contrary, many authors have argued, essentially on aesthetic grounds, that the form of the Maxwell equations suggests that magnetic charge should exist. In other words, no fundamental physical principle is known which can explain the apparent non-existence of a magnetic analogue of electric charge.

An ingenious and persuasive argument in favour of the existence of magnetic monopoles was put forward by Dirac† in 1931. The fact that the electric charge of all particles is an integral multiple of the electron charge (we ignore quarks) cannot be explained by Maxwell's theory; nor, however, can it be derived from the conventional formalism of the quantum theory. Dirac observed that if magnetic monopoles do exist the wave function - the fundamental mathematical object of the quantum theory - will not be single-valued unless a particular quantity is required to take on integral values. This proves to be possible only if the electric charge of any particle described by the wave function is an integral multiple of some basic charge. Thus, the existence of magnetic monopoles would allow us to explain an observed fact - the quantisation of charge - for which no other explanation is available. This constitutes strong circumstantial evidence in favour of the existence in favour of such particles. Dirac's argument is not affected by the extreme rarity of monopoles; neither the separations of the various particles, nor their abundances, enter the discussion at any stage. In the most extreme case, therefore. the existence of a *single* monopole would suffice to quantise the charge of every charged particle in the (casually accessible) universe.

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† P.A.M. Dirac, Proc. R. Soc. A 133,60 (1931).

In recent years, further circumstantial evidence in favour of the existence of magnetically charged particles has been uncovered, and this had led to an enormous increase in interest in the subject. In order to discuss these developments, we must consider *gauge theories*.

Maxwell's equations play suchan important role in the electro-magnetic theory that, in common parlance, the theory is often identified with the equations. The discovery of a set of equations governing the fields is a crucial step in the development of any "force-field" theory. By 1940, four apparently distinct types of forces were known: gravitation, electromagnetism, the "weak" forces (responsible for certain types of radioactive decay), and the "strong" force (responsible for holding nuclei together against their internal electrical repulsion). In only the first two cases, however, were field equations known, and this long remained a major obstacle in the path of attempts to gain a deeper understanding of the weak and strong interactions.

The field equations of electromagnetism have remarkably little in common with those of gravitation (Einstein's equations), a fact which never ceased to disturb Einstein himself. A noteworthy exception is the fact that both sets of equations exhibit a high degree of "symmetry". By this is meant that each set of equations is invariant under the action of some (Lie) group of transformations of its variables (not, however, the same group in each case). The basic idea of "gauge" theory - the name arises from irrelevant historical cicumstances - is to invert this observation: that is, instead of asking, "Given a set of field equations, what are the symmetries?" we ask, "Given a Lie group, what are the field equations possessing this symmetry?". Of course, we cannot expect a unique answer to such a question, but gauge theory does embody a particular procedure for answering it, and we shall argue that this prodedure is extremely natural from a mathematical point of view. Thus, one may conceive of gauge theory as an explicit "algorithm" which associates a definite set of G-invariant field equations with any Lie group G. In this way, we provide ourselves with a large array of internally consistent sets of field equations, and our hope is that the field equations of the weak and strong interactions may be found among this array.

In view of the fact that there are infinitely many different classical Lie groups, it may seem that gauge theory presents an embarrassment of riches and, to a certain extent, this is indeed the case. In practice, the method has worked extremely well. (In the following discussion, U(n) denotes the group n x n unitary matrices and SU(n) the subgroup of elements with unit determinant.) It now seems probable that the gauge equations corresponding to SU(3) constitute a suitable set of field equations for the strong interaction, and there is convincing evidence that the gauge equations arising from U(2) govern the weak and electromagnetic interactions simultaneously. This last theory provides a classic example of what is called a "gauge unification". The action of U(2) leaves the full set of field equations invariant, but mixes the weak and electromagnetic fields, so that the two interactions cannot be separated in an invariant fashion. In this very precise sense, we have a unified field theory. * The next step is to attempt to unify this "electroweak" interaction with the strong interaction, by considering a semi simple group which includes

* It is not widely known that Einstein was well aware of the importance of the "symmetry" concept in defining the term "unified field theory", at least a decade prior to the advent of gauge theory.

SU(3) x U(2) as a subgroup. Theories of this type unify three of the four known interactions (the inclusion of gravitation remains problematic) and are called "grand unified theories". A large class of such theories predict that the proton (which had previously been considered infinitely stable) should decay extremely slowly, and intensive experimental studies are at present being conducted to detect any such effect.

The relevance of all this to our topic derives from the fact that there is strong evidence to suggest that the field equations of most grand unified theories admit "magnetic monopole" solutions. (To be precise, the solutions in question represent stable concentrations of energy, which resemble magnetic monopoles when viewed from a large distance.) These monopoles possess various characteristics such as enormous masses, more than 10^{16} times that of the proton - which would identify them as "grand unified monopoles", so that their discovery would be regarded as concrete evidence in favour of grand unification. It has also been suggested that such monopoles could "catalyse" the decay of protons in their vicinity. (This process has (half seriously)been proposed as a future energy source.) For all these and other reasons, current interest in the experimental theoretical status of monopoles is intense. †

II

One of the most profound and fascinating features of gauge theory is the fact that it provides an example of the phenomenon called by Wigner "the unreasonable effectiveness of mathematics in the physical sciences". It is well known that the pure-mathematical work of Riemann (and his successors) on classical differential geometry provided Einstein with the basic framework for general relativity. Perhaps it is fitting, then, that modern differential goemetry stands in a similar relation to gauge theory. Almost the entire underlying mathematical framework of gauge theory is implicit in the work of Ehresmann, Chern and the other founders of modern differential geometry. Indeed, Wu and Yang have given a partial "dictionary"wherein all the basic concepts of gauge theory find a purely geometry counterpart. The existence of this geometric version of gauge theory, combined with general relativity theory, allows us to assert that all four fundamental forces of nature are (in some sense) *manifestations of geometry.*

This new formulation of gauge theory finds a particularly beautiful application in monopole theory. Let M denote the space-time manifold and let G be any Lie group. The product manifold M x G provides an example of what is known as a principal fibre bundle over M. In an obvious sense, M x G is a global product. A natural generalisation can be obtained if we consider a space P which is locally a product of M and G : that is, the part of P which lies "over" any open neighbourhood U in M is taken to be isomorphic to U X G. Such a generalised

† See the review article by C.A. Carrigan and W.P. Trower, Nature 305, 673 (1983).

product is called a non-trivial principal fibre bundle. Arbitrary principal fibre bundles over a paracompact M admit certain natural geometric structures called connections. By examining the kinds of connections which a particular principal bundle can admit, we can learn much about the topology of the bundle.

Upon examining the transformation behaviour of gauge fields under the action of the gauge group G, one finds that it is possible to identify gauge fields with connections in a principal fibre bundle which is locally a product of G with the space-time manifold M. It is possible to show that the local product structure can be extended to a gloabl product M x G if and only if M contains no magnetic monopoles. In this way, monopole theory acquires a deep mathematical interpretation. †

What is the value of this new formulation of gauge theory? The subject is not yet sufficient mature for a final assessment to be possible, but a number of motives can be given for developing it further. Firstly, the fire bundle formulation immediately renders it possible to state any gauge-theoretic problem in such a way that various extremely powerful techniques of differential geometry, algebraic topology, and so on, can be brought to bear on its solution. Thus the new approach is of value even from a purely technical point of view.

Secondly, and perhaps more importantly, experience has shown the way in which a theory is formulated can be absolutely decisive when we attempt to generalise that theory. It is most unlikely that general relativity theory could • have been constructed except on the basis of Minkowski's work on space-time: and yet Minkowski 'merely' reformulated special relativity in a mathematically opposite way. The fact that no plausible gauge theory which unifies all four interactions (grand unified theories exclude gravitation) has yet been proposed, probably indicates that a generalisation of gauge theory is called for. As we consider the problem of generalising gauge theory so as to construct a final synthesis of all known interactions, we are in much the same situation as Einstein when he began work on general relativity. It may not be idle to speculate that, as in the case of relativity, geometric ideas will play a fundamental role in the solution of this problem.

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†See the review paper of T. Eguchi, P.B. Gilkey, and A.J. Hanson, Phys. Rep. 66, 213(1980).